

CLASSIFICATION OF CONVEX ANCIENT SOLUTIONS TO CURVE SHORTENING FLOW ON THE SPHERE

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ABSTRACT. We prove that the only closed, embedded ancient solutions to the curve shortening flow on \mathbb{S}^2 are equators or shrinking circles, starting at an equator at time $t = -\infty$ and collapsing to the north pole at time $t = 0$. To obtain the result, we first prove a Harnack inequality for the curve shortening flow on the sphere. Then an application of the Gauss-Bonnet, easily allows us to obtain curvature bounds for ancient solutions leading to backwards smooth convergence to an equator. To complete the proof, we use an Aleksandrov reflection argument to show that maximal symmetry is preserved under the flow.

1. INTRODUCTION

In this paper we study a time-dependent family of smooth, embedded, closed curves $\gamma_t = F_t(\mathbb{S}^1) \subset \mathbb{S}^2$ on the unit sphere, evolving by the curve shortening flow:

$$(1) \quad \frac{\partial F_t}{\partial t} = -\kappa \mathbf{n}$$

where κ is the (signed) geodesic curvature of γ_t with respect to a choice of smooth unit normal vector field \mathbf{n} . Our aim is to classify convex ($\kappa > 0$) ancient solutions, which by definition exist on the maximal time interval $(-\infty, T)$. If $T \neq \infty$, we will assume from now on that $T = 0$. We prove the following theorem:

Theorem 1.1 (Classification of Ancient Solutions). *Let the family of curves γ_t be a closed, convex, embedded ancient solution to the curve shortening flow on the sphere S^2 . Then γ_t is either a fixed equator for all $t \in (-\infty, \infty)$, or a family of shrinking geodesic circles, existing on $(-\infty, 0)$, converging to an equator as $t \rightarrow -\infty$ and shrinking to a point at $T = 0$.*

The curve shortening flow has been studied extensively in the plane. The principal result is the Gage-Hamilton-Grayson theorem [GH86; Gra87], stating that arbitrary, smooth, embedded, closed solutions γ_t shrink to round points in finite time $T < \infty$. On surfaces, the curves shortening flow either

2010 *Mathematics Subject Classification.* 53C44, 35K55, 58J35.

Key words and phrases. Curve shortening flow, Ancient solutions, Aleksandrov reflection, Harnack.

collapses to a round point in finite time (as in the plane case), or exists for all time, converging to a closed geodesic as $t \rightarrow \infty$ [Gra89; Zhu98; JM10].

Ancient solutions to the curve shortening flow in the plane have been classified in [DHS10] as precisely the contracting circles and the contracting Angenent ovals. The former is a Type I ancient solution ($\lim_{t \rightarrow -\infty} \sup_{\gamma_t} |\kappa| < \infty$), while the latter is a Type II ancient solution ($\lim_{t \rightarrow -\infty} \sup_{\gamma_t} |\kappa| = \infty$). Our theorem shows that the only Type I ancient solution on the sphere are the "obvious" ones, and that no Type II ancient solutions exist.

To begin in section 2, we introduce some notation and preliminary results. Then in 3, we obtain a Harnack inequality for convex curves evolving by curve shortening on \mathbb{S}^2 in Theorem 3.1. This allows us to show that for convex curves, the curvature is monotonically increasing, hence bounded on any time interval $(-\infty, t_0]$. A standard bootstrapping argument then furnishes us with bounds on all higher derivatives. Next, in section 4 we use the Gauss-Bonnet theorem to show that $\int_{\gamma_t} \kappa \rightarrow 0$ as $t \rightarrow -\infty$. Combining this with the curvature estimates, we are readily able to show that γ_t converges smoothly to an equator as $t \rightarrow -\infty$. To complete the theorem, in section 5, we use a perturbed, parabolic version of Aleksandrov reflection inspired by [CG01; CG96]. This shows that γ_t reflects "above" (the precise definition is given in section 5) itself for the perturbed reflection and hence so too in the limit for all reflections preserving the equator. It is then easy to show that each γ_t is preserved under all equator preserving reflections and therefore is a round circle.

ACKNOWLEDGEMENTS

Both authors would like to thank Professor Bennett Chow for suggesting this problem and providing much useful guidance on laying out the program. The second author is especially thankful, this paper arising from her Ph.D. thesis under Professor Chow's supervision. This paper was completed while the first author was a SEW Visiting Assistant Professor at UCSD, acting as an informal Ph.D. advisor to the second author's Ph.D. research at UCSD.

2. NOTATION AND PRELIMINARIES

2.1. Convex Curves on \mathbb{S}^2 . A closed, embedded curve γ divides \mathbb{S}^2 into two open, disjoint regions. If one region has area strictly small than 2π , we label that region Ω^{int} and call it the interior of γ . The other region Ω^{ext} is the exterior. Let \mathbf{n} denote the interior pointing unit normal (so that for $x \in \gamma$, $\exp_x^{\mathbb{S}^2}(\epsilon \mathbf{n}) \in \Omega^{\text{int}}$ for small $\epsilon > 0$). Note that if the area of both regions equals 2π , it is not possible in general to single out one region as interior and one as exterior; consider for instance when γ is an equator. The issue is equivalent to defining a unit normal vector field on γ and declaring it be either interior or exterior pointing. In such a case, we will choose a unit normal vector field \mathbf{n} and designate it interior pointing.

On a Riemannian manifold M , there are several notions of convexity in common use. We will use the following definitions: A subset $K \subset M$ is *(geodesically) convex* if every two points $x, y \in K$ can be joined by a length minimising geodesic (of M) entirely contained within K . Note that we don't require this geodesic to be unique so that a closed hemisphere of \mathbb{S}^n is geodesically convex. K is *weakly (geodesically) convex* if any two points in K may be connected by a length minimizing geodesic of K . The difference between the two notions is that the length minimising geodesic in a weakly convex set need not be length minimising in M . It is well known that weakly convex is equivalent to non-negative boundary curvature. For example, if $M = \mathbb{S}^1 \times \mathbb{R}$ is a flat cylinder, then a geodesic disc of radius greater than $\pi/2$ is weakly convex, but not convex. On the sphere however (as in the plane), weakly convex sets are convex (Proposition 2.1). Since geodesics in \mathbb{S}^2 are great circles, and length minimising geodesics are half great circles, the length of any minimizing geodesic joining x to y is at most π . The following proposition characterises convex regions K of \mathbb{S}^2 with boundary a smooth embedded curve. The results (and arguments) are standard and well known, but we could not find a good single reference, so give the details here.

Proposition 2.1. *Let $K \subset \mathbb{S}^2$ be a connected open set with boundary $\partial K = \gamma$ a smooth, closed embedded curve. Let \mathbf{n} be the interior unit normal vector field along γ and κ the geodesic curvature with respect to \mathbf{n} . Then the following are equivalent:*

- (1) K is convex,
- (2) for every $x \in \gamma$, $K \subset H_x^+$ where H_x^+ is the hemisphere with boundary the tangent great circle E_x to $\gamma(x)$ and interior normal $\mathbf{n}(x)$,
- (3) γ may be written as the graph over an equator of a smooth function f such that

$$f''(\theta) + 2 \tan(f(\theta))(f'(\theta))^2 + \cos(f(\theta)) \sin(f(\theta)) \geq 0, \quad \theta \in \mathbb{S}^1.$$
- (4) The boundary curvature, $\kappa \geq 0$.

Proof. • (1) \Rightarrow (2):

Let $x \in \gamma$ and let E_x be the tangent great circle to $\gamma = \partial K$ at x . With $\mathbf{n}(x)$ pointing interior to K , let $H^+(x)$ denote the hemisphere with boundary E_x and interior normal $\mathbf{n}(x)$. Then $H^+(x) \cap K \neq \emptyset$ and we need to show that in fact $K \subset H^+(x)$.

Suppose otherwise, so that there is a y in the interior of H^- , and let σ be the unique (since $y \notin E_x$) length minimising geodesic joining x to y . σ lies in H^- , but also does not intersect K in a neighbourhood of x since it points exterior to K ($\langle \sigma'(x), \mathbf{n}(x) \rangle < 0$). On the other hand, since K is convex and both $x, y \in K$ we must have $\sigma \subset K$, a contradiction.

- (2) \Rightarrow (3):

We work in polar coordinates $(\cos \phi \cos \theta, \cos \phi \sin \theta, \sin \phi)$ with the equator given by $\phi = 0$ (as opposed to the usual convention of

$\phi = 0$ at the north pole). For graphs $(\theta, f(\theta))$ the curvature is given by

$$\kappa = \frac{\cos(f)}{((f')^2 + \cos^2(f))^{3/2}} (f'' + 2(f')^2 \tan(f) + \sin(f) \cos(f)).$$

So here, we prove that (2) implies γ may be written as a graph and that $\kappa \geq 0$.

γ is a graph: Take any tangent great circle E , H^+ the hemisphere with $K \subset H^+$, and p the center of H^+ . If $p \in \gamma$, then we can rotate E along the geodesic joining p to x to obtain a new E with $K \subset H^+$ and $p \in K$ (remember K is open). Then by convexity, for any $y \in E$, the geodesic ray joining p to y intersects γ in precisely one point, which we may denote by $f(y)$, expressing γ as the graph of f (which must be smooth).

$\kappa \geq 0$: This follows since K lies on one side of every tangent great circle, so the local Taylor expansion of γ shows that the curvature vector is interior pointing everywhere.

- (3) \Rightarrow (4):

As above, the condition on f is precisely that $\kappa \geq 0$.

- (4) \Rightarrow (1):

We prove the contrapositive. Suppose that K is not convex. We need to show that there is a $y \in \gamma$ such that $\kappa(y) < 0$.

Since K is not convex, there is a length minimising geodesic α meeting K^- only at its endpoints. Moreover, we can choose α to have length strictly less than π : if not, then any arbitrary great circle must intersect K^C in a connected arc of length at least π and so intersects K in a connected arc of length at most π . Therefore any $x, y \in K$ may be connected by a length minimising geodesic contradicting that K is not convex.

Let σ_z be the continuous family of geodesic rays of length π starting at $z \in \alpha$, perpendicular to α . There are precisely two such families, and we choose σ_z so that σ_z intersects γ at distance less than π for z near the endpoints of α . Note that on the sphere, for each $z \in \alpha$, we have $\sigma_z \cap \gamma \neq \emptyset$ since the end points of α (which also lie on γ) lie on either side of the equator containing σ_z . Let $y = y(z) \in \gamma$ be the first point where γ_z meets γ and let $\rho(z) = d(z, y)$.

Then ρ is continuous and attains its maximum at a point z_0 in the interior of α since $\rho = 0$ on the end points of α and by assumption $\rho(z) > 0$ for some $z \in \alpha$. Now we have ρ_{z_0} a geodesic meeting γ orthogonally at y_0 , hence we can solve for z as a function of y near y_0 . Then the second variation formula (varying y) shows that $\kappa(y_0) \leq 0$.

□

From here on we will freely use the results of the proposition without further comment and by a *convex curve* we will mean a closed, embedded

curve γ satisfying any of the four conditions. Lastly, let us note that for γ convex, approximating γ by convex polygons (with geodesic arcs), it is possible to show that the length $L(\gamma) \leq 2\pi$, a fact we will employ in section 4. See [Top06, Problem 1.10.1] for details. In section 5, we will find it very useful to write γ as the graph over an equator.

2.2. Evolution of basic quantities. Let us now record the evolution of various quantities under the curve shortening flow. This is very similar to the plane case [GH86; Gag84]. We make use of the Serret-Frenet formulae,

$$\nabla_{\mathbf{t}} \mathbf{t} = \kappa \mathbf{n}, \quad \nabla_{\mathbf{t}} \mathbf{n} = -\kappa \mathbf{t}$$

with \mathbf{t} the unit tangent to γ and \mathbf{n} the interior pointing normal.

Let $s = s_t$ denote the arc-length parameter of γ_t . The commutator of $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ is

$$(2) \quad \left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right] = -\kappa^2 \frac{\partial}{\partial s}.$$

Under the curve shortening flow on \mathbb{S}^2 , the curvature evolves according to

$$(3) \quad \kappa_t = \kappa_{ss} + \kappa^3 + \kappa$$

where subscripts denote partial derivatives. The maximum principle now ensures that if $\kappa > 0$ at some time t_0 , then this holds for all $t \geq t_0$.

Lastly, the element of arc-length ds evolves according to

$$(4) \quad \frac{\partial}{\partial t} ds = -\kappa^2 ds.$$

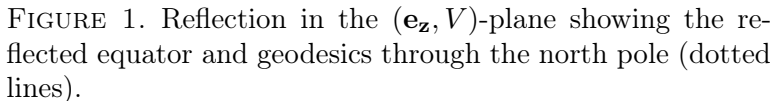
2.3. Aleksandrov reflection. In section 5 we will make use of an Aleksandrov reflection argument, so give the preliminaries here. Embed \mathbb{S}^2 in \mathbb{R}^3 via the standard embedding, and let E denote the equator $\{z = 0\}$.

The argument rests on a "tilted" Aleksandrov reflection. Let \mathbf{V} be a vector in \mathbb{R}^3 such that $\langle \mathbf{V}, \mathbf{e}_z \rangle < 0$ where $\mathbf{e}_z = (0, 0, 1)$, and let $P_{\mathbf{V}}$ be the plane through the origin with normal vector \mathbf{V} . Then $P_{\mathbf{V}}$ intersects E in two antipodal points. Let $\delta(V) \in (0, \pi/2)$ be the angle between \mathbf{V} and the plane $\{z = 0\}$. Notice that for each fixed δ , the set of \mathbf{V} with $\delta(\mathbf{V}) = \delta$ is a compact set parameterised by \mathbb{S}^1 acting as rotations about the z -axis. We consider the Aleksandrov reflection across the plane P ,

$$R_{\mathbf{V}}(X) = X - 2\langle X, \mathbf{V} \rangle \mathbf{V}$$

which is an isometry of \mathbb{R}^3 preserving \mathbb{S}^2 , hence is also an isometry of \mathbb{S}^2 . See figure 1.

Let $H_{\mathbf{V}}^+ = \{X \in \mathbb{R}^3 : \langle X, \mathbf{V} \rangle > 0\}$ be the open half space lying on the side of $P_{\mathbf{V}}$ into which \mathbf{V} points, and $H_{\mathbf{V}}^- = \{X \in \mathbb{R}^3 : \langle X, \mathbf{V} \rangle < 0\}$ be the open half space on the other side of $P_{\mathbf{V}}$. For any set $S \subset \mathbb{R}^3$, let $S_{\mathbf{V}}^{\pm} = S \cap H_{\mathbf{V}}^{\pm}$. In particular $(\mathbb{S}^2)_{\mathbf{V}}^{\pm}$ are hemispheres with boundary equator equal to $P_{\mathbf{V}} \cap \mathbb{S}^2$. Notice that $R_{\mathbf{V}}$ takes $H_{\mathbf{V}}^{\pm}$ to $H_{\mathbf{V}}^{\mp}$.



For any two curves α, β on \mathbb{S}^2 , and any $X \in E$, let us write $\alpha \geq_X \beta$ (resp. $\alpha >_X \beta$) if

$$\inf\{\rho(Y) : Y \in \pi^{-1}(X) \cap \alpha\} \geq (\text{resp. } >) \sup\{\rho(Y) : Y \in \pi^{-1}(X) \cap \beta\}$$

whenever both sets are non-empty. The inf and sup are required since α and β need not be graphs over the equator and so the fibres $\pi^{-1}(X) \cap \alpha$ and $\pi^{-1}(X) \cap \beta$ may have multiple points. Loosely speaking, we say α lies above β over the point X in the equator $\{z = 0\}$. We will also write $\alpha \geq$ (resp. $>$) β if $\alpha \geq_X$ (resp. $>_X$) β for every $X \in E$. Notice in particular that we require strict inequality to hold for *every* X .

Remark 2.2. The relations \leq_X and \leq are not partial orders in general since they are not reflexive. In fact, $\alpha \leq_X \alpha$ if and only if the fibre $\pi^{-1}(X)$ intersects α in a single point. The relation \leq is only a partial order when restricted to curves that are graphs over the equator: $\alpha \leq \alpha$ if and only if α is a graph over the equator.

Using polar coordinates as above, we may also rewrite $\alpha \geq_X \beta$ if and only if $\theta(X) \in \{\theta(\alpha)\} \cap \{\theta(\beta)\}$ and

$$\inf\{\phi(\alpha) : \theta(\alpha) = \theta(X)\} \geq \sup\{\phi(\beta) : \theta(\beta) = \theta(X)\}.$$

That is, $\alpha \geq_X \beta$ if and only if there is at least one point on α and at least one point on β with azimuthal angle θ equal to the azimuthal angle of X and so that the smallest polar angle ϕ of α is greater than or equal to the greatest polar angle of β .

3. HARNACK INEQUALITY AND CURVATURE ESTIMATES

Just as for the curve shortening flow in the plane, there is a Harnack inequality for the curve shortening flow on \mathbb{S}^2 . This is the fundamental result of this section, from which everything else follows.

Theorem 3.1 (Harnack Inequality). *For any immersed solution to the curve shortening flow defined on the time interval $[-\alpha, 0)$ and with $\kappa > 0$, we have*

$$(\log k)_{ss} + k^2 + \frac{1}{2(t - \alpha)} \geq 0.$$

Proof. From the evolution of the curvature in equation (3) and the commutator equation (2), we deduce

$$\begin{aligned} k_{st} &= k_{ts} + k^2 k_s = (k_{sss} + 3k^2 k_s + k_s) + k^2 k_s \\ &= k_{sss} + 4k^2 k_s + k_s \\ k_{sst} &= k_{tss} + 2k k_s^2 + 2k^2 k_{ss} = (k_{ssss} + 3k^2 k_{ss} + 6k k_s^2 + k_{ss}) + 2k k_s^2 + 2k^2 k_{ss} \\ &= k_{ssss} + 5k^2 k_{ss} + 8k k_s^2 + k_{ss}. \end{aligned}$$

Let Q be the quantity

$$Q = (\log k)_{ss} + k^2.$$

Computing the time derivative, we get

$$\begin{aligned}
Q_t &= -\frac{k_{ss}}{k^2}k_t + \frac{k_{sst}}{k} + 2k^{-3}k_t k_s^2 - k^{-2}(2k_s k_{st}) + 2kk_t \\
&= -\frac{k_{ss}^2}{k^2} + 6kk_{ss} + \frac{k_{ssss}}{k} + 2k_s^2 + \frac{2k_s^2 k_{ss}}{k^3} - \frac{2k_s k_{sss}}{k^2} + 2k^4 + 2k^2 \\
&= Q_{ss} + \left(\frac{2k_s}{k}\right)Q_s + 2Q^2 + 2k^2 \\
&\geq Q_{ss} + \left(\frac{2k_s}{k}\right)Q_s + 2Q^2.
\end{aligned}$$

An ODE comparison with $q(t) = -1/2(t - \alpha)$ which satisfies $q_t = 2q^2$ and $\lim_{t \rightarrow \alpha} q(t) = -\infty$ shows that $Q(s, t) \geq q(t)$ completing the result. \square

Corollary 3.2. *For an ancient solution γ_t , with $\kappa > 0$, we have*

$$\kappa_t \geq 0.$$

Proof. As γ_t is an ancient solution with $\kappa > 0$, the Harnack inequality holds for any $\alpha < 0$, supplying us with

$$\begin{aligned}
0 &\leq (\log k)_{ss} + k^2 + \frac{1}{2(1 - \alpha)} \\
&= \frac{\kappa_{ss}}{\kappa} - \frac{\kappa_s^2}{\kappa} + \kappa^2 + \frac{1}{2(1 - \alpha)} \\
&\leq \frac{\kappa_{ss} + \kappa^3 + \kappa}{\kappa} + \frac{1}{2(1 - \alpha)} \\
&= \frac{\kappa_t}{\kappa} + \frac{1}{2(1 - \alpha)}
\end{aligned}$$

for $t \in [\alpha, 0)$. Taking the limit $\alpha \rightarrow -\infty$ gives the result. \square

We are now able to obtain a curvature bound, and by standard bootstrapping arguments, we also obtain higher derivative bounds.

Corollary 3.3. *For any $t_0 < 0$ and any integer $j \geq 0$, there exists a constant $C_j(t_0)$ such that*

$$|\kappa^{(j)}| \leq C_j(t_0)$$

on $(-\infty, t_0)$.

Proof. By Corollary 3.2, κ is increasing in t , and since $\kappa > 0$, we may take $C_0(t_0) = \sup\{\kappa(x, t_0) : x \in \mathbb{S}^1\}$.

The higher derivative estimates follow by standard bootstrapping arguments similar to those described in [Ham95]. For example, we obtain $C_1(t_0)$ from the evolution equation $(k_s)_t = k_{sss} + 4k^2 k_s + k_s$ by applying the maximum principle to the evolution of $(t - (t_0 - 1))k_s^2 + C_0(t_0)k^2$ and using the fact that $|\kappa| \leq C_0(t_0)$. \square

4. BACKWARDS CONVERGENCE

Armed with the curvature bounds, we can prove that any ancient solution γ_t converges smoothly to an equator as $t \rightarrow -\infty$. We begin with a lemma.

Lemma 4.1. *Let γ_t be an ancient solution to the curve shortening flow. Then*

$$\lim_{t \rightarrow -\infty} \int_{\gamma_t} k ds = 0$$

exponentially fast.

Proof. By the Gauss-Bonnet theorem we have

$$\int_{\gamma_t} \kappa ds = 2\pi - A$$

where A is the area of Ω_t^{int} . Recalling that $\kappa_t = \kappa_{ss} + \kappa^3 + \kappa$ and that $(ds)_t = -\kappa^2 ds$, we obtain

$$\frac{\partial}{\partial t} \int_{\gamma_t} \kappa ds = \int_{\gamma_t} \kappa_{ss} + \kappa ds = \int_{\gamma_t} \kappa ds = 2\pi - A.$$

Therefore

$$A_t = A - 2\pi,$$

and hence

$$(5) \quad A = 2\pi[1 - (1 - A(0)/2\pi)e^t].$$

Then letting $t \rightarrow -\infty$, we find that $A(t) \rightarrow 2\pi$ exponentially fast and the Gauss-Bonnet formula implies that

$$\int_{\gamma_t} \kappa ds \rightarrow 0$$

exponentially fast. \square

Combining our estimates so far, we now obtain the following important proposition:

Proposition 4.2. *Let γ_t be an ancient solution to the curve shortening flow. Then for every integer $j \geq 0$, we have*

$$\max_{s \in \mathbb{S}^1} |\kappa^{(j)}|(s, t) \rightarrow 0$$

as $t \rightarrow -\infty$.

Proof. First, let us prove the case $j = 0$. We argue by contradiction. Suppose the proposition is false. Then there exists $\epsilon > 0$, a sequence $t_i \rightarrow -\infty$ and, a sequence s_i such that $\kappa(s_i, t_i) \geq \epsilon$ for all i . Since we also have $|\kappa_s| \leq C_1(1)$ for $t \leq 1$, we find that

$$\kappa(s, t_i) \geq \epsilon - C_1(1)|s - s_i| \geq \epsilon/2$$

for all s such that $|s - s_i| \leq \frac{\epsilon}{2C_1(1)}$. But this implies that for every i ,

$$\int_{\gamma_{t_i}} \kappa(s, t) ds \geq \int_{|s-s_i| \leq \frac{\epsilon}{2C_1(1)}} \kappa(s, t) ds \geq \frac{\epsilon^2}{4C_1(1)}$$

contradicting the fact that $\int_{\gamma_t} \kappa \rightarrow 0$ as $t \rightarrow -\infty$ by lemma 4.1.

The result for $j > 0$ follows by a bootstrapping argument similar to the proof of Corollary 3.3. \square

Now we have all the ingredients to prove that the backwards limit is an equator. First we have sub-sequential convergence.

Lemma 4.3. *Let γ_t be an ancient, embedded, convex solution to the curve shortening flow on \mathbb{S}^2 . Then there is a sequence $t_k \rightarrow -\infty$ with $\gamma_{t_k} \rightarrow_{C^\infty} \gamma_{-\infty}$ as $k \rightarrow \infty$, with $\gamma_{-\infty}$ an equator.*

Proof. Since $\frac{\partial}{\partial t} L = -\int \kappa^2 ds < 0$, $L(t)$ is bounded below by $L(-1) > 0$ for all $t \leq -1$. By the paragraph following Proposition 2.1, $L \leq 2\pi$.

Proposition 4.2 shows that the curvature and all derivatives converge to 0. Since L is bounded, as $t \rightarrow -\infty$, $|\gamma'|$ is bounded (say in a constant speed parametrisation) above and away from zero. The Arzela-Ascoli theorem then provides us with a sequence $t_k \rightarrow -\infty$ such that γ_{t_k} converges smoothly to a closed, immersed curve with zero curvature, i.e. to an equator $\gamma_{-\infty}$. \square

Next we show that the limit is unique and that the flow remains in a fixed hemisphere.

Corollary 4.4. *The equator $\gamma_{-\infty}$ is unique and γ_t lies in one of the hemispheres $H_{-\infty}^\pm$ defined by $\gamma_{-\infty}$ for all $t \in (-\infty, 0)$.*

Proof. Since $\kappa > 0$, Gauss-Bonnet implies that $A(t) < 2\pi$ for all t and that the curvature vector points inward (recall that the interior Ω_t^{int} is the region enclosed by γ_t with area less than 2π). Thus $\Omega_{t_1}^{\text{int}} \subsetneq \Omega_{t_2}^{\text{int}}$ whenever $t_2 < t_1$.

In particular, for our sequence (t_k) , $\Omega_{t_j}^{\text{int}} \subsetneq \Omega_{t_k}^{\text{int}}$ for $k > j$. If $\Omega_{t_j}^{\text{int}}$ is not wholly contained in either hemisphere $H_{-\infty}^\pm$ for some j , then it contains points in both hemispheres, $x_j^\pm \in H_{-\infty}^\pm$. We obtain a contradiction by choosing $k < j$ with γ_{t_k} sufficiently close to $\gamma_{-\infty}$ so that x_j^+ lies on the opposite side of $\gamma_{-\infty}$ to x_j^- contradicting $\Omega_{t_k}^{\text{int}} \subset \Omega_{t_j}^{\text{int}}$ has points on both sides of γ_{t_k} . Thus $\Omega_{t_k}^{\text{int}}$ lies entirely in one or the other hemisphere $H_{-\infty}^\pm$ for every k .

Now for any $t \in (-\infty, 0)$, choose k such that $t_k < t$. Then $\Omega_t^{\text{int}} \subsetneq \Omega_{t_k}^{\text{int}}$, the latter lying in a hemisphere. Lastly, suppose there is a sequence t'_k with $\gamma_{t'_k}$ converging to different equator. This equator has points lying in both hemispheres defined by $\gamma_{-\infty}$ and hence $\gamma_{t'_k}$ also has points in both hemispheres for t'_k sufficiently negative, a contradiction. \square

Remark 4.5. Any closed, embedded curve on \mathbb{S}^2 with $\kappa \geq 0$ must lie in a closed hemisphere. The result above shows that under the flow, an embedded, convex, ancient solution remains in a fixed hemisphere for all time.

Now we can extend the sub-sequential convergence to full convergence.

Theorem 4.6. *Let γ_t be an ancient, embedded, convex solution to the curve shortening flow on \mathbb{S}^2 . Then $\gamma_t \rightarrow_{C^\infty} \gamma_{-\infty}$ up to diffeomorphism as $t \rightarrow -\infty$.*

Proof. First, we have C^0 convergence: for any $\epsilon > 0$, choose t_k with γ_{t_k} ϵ -close to $\gamma_{-\infty}$ in C^0 norm. Then for any $t < t_k$, both γ_t and γ_{t_k} lie in the same hemisphere with $\Omega_{t_k}^{\text{int}} \subsetneq \Omega_t^{\text{int}}$. Thus γ_t lies between $\gamma_{-\infty}$ and γ_{t_k} and so is also ϵ -close to $\gamma_{-\infty}$ in C^0 norm.

C^1 convergence follows since the total length $L(t_k) \rightarrow 2\pi$ as $k \rightarrow \infty$. But also $\partial L / \partial t = -\int k^2 ds < 0$ so that L is monotone increasing backwards in time hence $L(t) \rightarrow 2\pi$ as $t \rightarrow -\infty$. But now parametrising γ_t on $[0, 1]$ with constant speed gives $|\gamma'_t| = L(t) \rightarrow 2\pi$ and so C^1 convergence up to diffeomorphism follows.

Smooth convergence up to diffeomorphism now follows since the curvature and all derivatives of curvature converge to 0. \square

5. ANCIENT SOLUTIONS ARE SHRINKING ROUND CIRCLES

In this section, we prove that ancient, convex solutions to the curve shortening flow are shrinking round circles.

Lemma 5.1 (Backwards approximate symmetry). *For any $\delta \in (0, \pi/2)$, there exists a $t_\delta \in (-\infty, 0)$ such that for every V with $\delta(V) = \delta$ and all $t \leq t_\delta$, we have $R_{\mathbf{V}}((\gamma_t)^+_{\mathbf{V}}) \geq (\gamma_t)^-_{\mathbf{V}}$.*

Proof. We use polar coordinates as in section 2. From Corollary 4.4 and Proposition 2.1, we can assume that on $(-\infty, 0)$, γ_t lies in the upper hemisphere $\{z > 0\}$, written as a graph $\phi = f_t(\theta)$ of a smooth family of positive, smooth functions $f_t[0, 2\pi] \rightarrow \mathbb{R}$. Since γ_t smoothly converges uniformly to the equator $\{z = 0\}$, we have $\frac{\partial^k}{\partial \theta^k} f_t \rightarrow 0$ uniformly for each $k \geq 0$.

Provided that $\delta < \pi/4$, the reflected equator $R_{\mathbf{V}}(\{z = 0\})$ can be written as a graph $(\theta, g_{-\infty}(\theta))$. Since $R_{\mathbf{V}}$ is an isometry, $R_{\mathbf{V}}(\gamma_t)$ converges smoothly and uniformly to the reflected equator $R_{\mathbf{V}}(\{z = 0\})$. As the latter is a graph, possibly by choosing $t_0 < 0$ independently of δ , we can assume that $R_{\mathbf{V}}(\gamma_t)$ may be written as a graph $(\theta, g_t(\theta))$ for $t < t_0$ with $g_t \rightarrow g_{-\infty}$ smoothly as $t \rightarrow -\infty$.

In spherical polar coordinates, for $X, Y \in \mathbb{S}^2$ the nearest-point projection is $(\theta(X), \phi(X)) \mapsto (\theta(X), 0)$. If $\theta(X) = \theta(Y)$, the statement $X \geq Y$ is equivalent to $\phi(X) \geq \phi(Y)$. Thus to show that $R_{\mathbf{V}}(\gamma_t)^+ \geq \gamma_t^-$ it is enough to show that $g_t(\theta) \geq f_t(\theta)$.

The proof is composed of estimates for *interior points* (i.e. points away from $P \cap \mathbb{S}^2$) and for *boundary points* (i.e. points near $P \cap \gamma_t$).

Interior Points

For $\delta < \pi/4$, the reflected equator $(\theta, g_{-\infty}(\theta))$, $\theta \in [0, \pi]$ is given by a non-negative, smooth, concave function $g_{-\infty}$ symmetric about $\pi/2$ and strictly positive for $\theta \in (0, \pi)$. Given any $\epsilon \in (0, \pi/2)$, let $G = g_{-\infty}(\epsilon) = g_{-\infty}(\pi - \epsilon)$.

Then $G < g_\infty(\theta)$ for any $\theta \in (\epsilon, \pi - \epsilon)$. Choose $t_1 < t_0$ such that for $\theta \in (\epsilon, \pi - \epsilon)$ and $t < t_1$, we have $f_t(\theta) < G/2$ and $|g_t(\theta) - g_{-\infty}(\theta)| < G/2$. This is possible since $f_t \rightarrow 0$ uniformly, and $g_t \rightarrow g_{-\infty}$ uniformly. Then, since $g_{-\infty} > G$ on $(\epsilon, \pi - \epsilon)$, for any $\epsilon > 0$, there is a $t_1 = t_1(\epsilon)$ such that $g_t(\theta) > f_t(\theta)$ for $\theta \in (\epsilon, \pi - \epsilon)$ and $t \leq t_1$.

Boundary Points

Choose an orientation on θ so that $P \cap \{z > 0\}$ lies in the region with $\theta \in (-\pi, 0)$. Then recalling that γ_t is a graph over the equator, $\gamma_t \cap P = \{\theta_0(t), \theta_1(t)\}$ with $\theta_0(t) \in (-\pi, 0)$ and $\theta_1(t) \in (\pi, 2\pi)$. Moreover as $t \rightarrow -\infty$ we have $\theta_0(t) \rightarrow 0$ and $\theta_1(t) \rightarrow \pi$. The aim is to show that given $\tilde{\epsilon} > 0$, there is a $t_{\tilde{\epsilon}}$ such that $g_t > f_t$ on $(\theta_0(t), \tilde{\epsilon}) \cup (\pi - \tilde{\epsilon}, \theta_1(t))$ for every $t < t_{\tilde{\epsilon}}$. It's enough to prove it on $(\theta_0(t), \tilde{\epsilon})$. The proof on $(\pi - \tilde{\epsilon}, \theta_1(t))$ is similar.

We use that $f_t \rightarrow 0$, and $g_t \rightarrow g_{-\infty}$ smoothly and uniformly, and that $\theta_0(t) \rightarrow 0$ as $t \rightarrow -\infty$. Notice that $R_{\mathbf{V}}(\{z = 0\})$ lies above the equator $\{z = 0\}$ for $\theta \in (0, \pi)$ and lies below for $\theta \in (-\pi, 0)$. Thus, $g_{-\infty}$ is odd about $\theta = 0$ and increasing near $\theta = 0$ so that $g'_{-\infty}(0) > 0$ (in fact equal to $\tan(2\delta)$) and $g''_{-\infty}(0) = 0$.

Choose $t_1 < t_0$ such that $g'_t(\theta_0(t)) > f'_t(\theta_0(t))$ for all $t < t_1$ which we can do since $f'_t \rightarrow 0$ uniformly and $g'_t(\theta_0) \rightarrow g'_{-\infty}(0) = \tan(2\delta) > 0$. We also have that $g_t(\theta_0) = f_t(\theta_0)$ since θ_0 is the point about which f_t is reflected across P . Expand g_t and f_t in a Taylor series about $\theta_0(t)$ to get that for $\theta > \theta_0$, $g_t > f_t$ if and only if

$$g'_t(\theta_0) - f'_t(\theta_0) > -\frac{1}{2}(g''_t(c) - f''_t(c))(\theta - \theta_0)$$

where $c = c(\theta, t) \in (\theta_0(t), \theta)$. As $t \rightarrow -\infty$ the left hand side converges to $\tan(2\delta)$ whilst the right hand side converges to 0 hence there is a $t_2 = t_2(\tilde{\epsilon}) < t_1$ such that the inequality $g_t - f_t > 0$ is satisfied for any $\theta \in (\theta_0(t), \tilde{\epsilon})$ and $t < t_2$.

Combined estimates

To finish the proof, fix any $\epsilon > 0$ and use the interior estimates to obtain $g_t > f_t$ for $\theta \in (-\epsilon, \pi - \epsilon)$ and $t < t_1$. Then choose $\tilde{\epsilon} > \epsilon$ to obtain $g_t > f_t$ for $\theta \in (\theta_0(t), \tilde{\epsilon}) \cup (\pi - \tilde{\epsilon}, \theta_1(t))$ and $t < t_2$ from the boundary estimates. Then let $t_\delta = \min\{t_1, t_2\}$ to get $g_t > f_t$ for all $\theta \in (\theta_0(t), \theta_1(t))$ and all $t < t_\delta$. \square

Lemma 5.2 (Approximate symmetry preserved). *There is a $T \in (-\infty, 0)$ such that $R_{\mathbf{V}}(\gamma_t)^+ \geq \gamma_t^-$ for $t \in (-\infty, T)$ and all $\delta \in (0, \pi/4)$.*

Proof. Recall that both the γ_t^- and $R_{\mathbf{V}}(\gamma_t)^+$ may be written as graphs over the equator for $t \in (-\infty, t_0)$. Since both the equator $\gamma_{-\infty}$ and the reflected equator $R_{\mathbf{V}}(\gamma_{-\infty})$ meet P transversely, γ_t smoothly converges to the equator, and $R_{\mathbf{V}}$ is an isometry, there is a $t_1 = t_1(\delta) \in (-\infty, t_0)$ such that both γ_t^- and $R_{\mathbf{V}}(\gamma_t)^+$ meet P transversely for all $t \in (-\infty, t_1)$. Thus γ_t^- and $R_{\mathbf{V}}(\gamma_t)^+$ are connected curves meeting P transversely in precisely two points for each t .

Now we apply the maximum principle. Since $R_{\mathbf{V}}$ is an isometry, $R_{\mathbf{V}}(\gamma_t^+)$ evolves by curve shortening. Since $P \cap \mathbb{S}^2$ is a great circle, it is stationary

under the curve shortening flow so we can think of it too evolving by curve shortening. Therefore, as both γ_t^- and $R_{\mathbf{V}}(\gamma_t)^+$ meet P transversely, the maximum principle ensures that both curves do not intersect P at any *other* points, hence remain in $\mathbb{S}^2_{\mathbf{V}}^-$ for all $t \in (-\infty, t_1)$.

The above allows us to set up a maximum principle argument: we have two connected curves γ_t^- , $R_{\mathbf{V}}(\gamma_t)^+$ evolving by curve shortening and they agree at their end points which remain on P . By Lemma 5.1, for all $t \in (-\infty, t_\delta)$ we have $d(x, y, t) > 0$ for any $x \in \gamma_t^-$ and $y \in R_{\mathbf{V}}(\gamma_t)^+$ away from the end points. We also obtain that at the end points, the angle $R_{\mathbf{V}}(\gamma_t)^+$ makes with the $\{z = 0\}$ plane is strictly bigger than the angle γ_t^- makes with the $\{z = 0\}$ plane. By the parabolic Hopf boundary point lemma (see e.g. [Cho97]), this positive lower bound is preserved under the flow and so $d(x, y, t) > 0$ for (x, y) near both end points. Now, in the usual way (e.g. [Hui98]) a contradiction is obtained if $d(x, y, t) \leq 0$ at some time t for some (x, y) since this must occur at a first time $t > t_\delta$ at an interior point (x, y) .

This furnishes us with a T_δ for each δ such that $R_{\mathbf{V}}(\gamma_t)^+ \geq \gamma_t^-$ for $t \in (-\infty, T_\delta)$. Let $T = \inf\{T_\delta : \delta \in (0, \pi/2)\}$. We need to show that $T > -\infty$. To see this, observe that the above argument is valid provided both γ_t^- and $R_{\mathbf{V}}(\gamma_t)^+$

- (1) are graphs over the equator,
- (2) meet P transversely,
- (3) are non-empty

for $t \in (-\infty, T)$.

- (1) Recall that $R_{\mathbf{V}}(\gamma_{-\infty})$ is a graph $g_{-\infty}^\delta$ for each $\delta \in (0, \pi/4)$ with maximum derivative at the end points $\theta = \{0, \pi\}$. As $\delta \rightarrow 0$, the derivative $(g_{-\infty}^\delta)'(0)$ monotonically decreases to 0. For each fixed t then $[R_{\mathbf{V}}(\gamma_t)^+]'$ becomes more horizontal as δ decreases hence if $R_{\mathbf{V}}(\gamma_t)^+$ is a graph (so does not have a vertical tangent) for some δ_0 , then it remains a graph for every $\delta < \delta_0$. Of course whether γ_t^- is a graph or not is independent of δ , and by convexity we know that γ_t is a graph for all $t \in (-\infty, 0)$.
- (2) The angle P makes with the $\{z = 0\}$ plane increases monotonically as $\delta \rightarrow 0$. If γ_t^- and $R_{\mathbf{V}}(\gamma_t)^+$ meet P transversely for some δ_0 then they continue to do so for every $\delta < \delta_0$.
- (3) Provided γ_t^- lies below the maximum ϕ coordinate of $P \cap \mathbb{S}^2$, both curves γ_t^- and $R_{\mathbf{V}}(\gamma_t)^+$ are non-empty. But now just observe that this ϕ monotonically increases to $\pi/2$ as $\delta \rightarrow 0$.

□

Next we characterise those curves α with maximal approximate symmetry for every $\delta > 0$ as round circles.

Proposition 5.3 (Exact symmetry). *Let $\alpha \subset (\mathbb{S}^2)^+ = \mathbb{S}^2 \cap \{z \geq 0\}$ be a smooth curve. If $R_{\mathbf{V}}(\alpha_{\mathbf{V}}^+) \geq \alpha_{\mathbf{V}}^-$ for every \mathbf{V} such that $\langle \mathbf{V}, \mathbf{e}_z \rangle < 0$ and $\delta(\mathbf{V}) \in (0, \pi/4)$, then α is a round circle with center the north pole $(0, 0, 1)$.*

Proof. Let \mathbf{V}_0 be a vector in \mathbb{R}^3 such that $\langle \mathbf{V}_0, \mathbf{e}_z \rangle = 0$ (so that $\delta(\mathbf{V}_0) = 0$). Choose any $X \in (\gamma_{-\infty})_{\mathbf{V}}^-$. Then by assumption, we have

$$R_{\mathbf{V}}(\alpha_{\mathbf{V}}^+) \geq_X \alpha_{\mathbf{V}}^-$$

for all \mathbf{V} lying in the plane spanned by \mathbf{e}_z and \mathbf{V}_0 , and with $\langle \mathbf{V}, \mathbf{e}_z \rangle < 0$ and $\delta(\mathbf{V}) \in (0, \pi/4)$. By continuity, letting $\mathbf{V} \rightarrow \mathbf{V}_0$ we obtain $R_{\mathbf{V}_0}(\alpha_{\mathbf{V}_0}^+) \geq_X \alpha_{\mathbf{V}_0}^-$ for each $X \in \gamma_{-\infty}^-$ and hence

$$R_{\mathbf{V}_0}(\alpha_{\mathbf{V}_0}^+) \geq \alpha_{\mathbf{V}_0}^-$$

for every \mathbf{V}_0 with $\langle \mathbf{V}_0, \mathbf{e}_z \rangle = 0$.

Now, we need some simple properties of $R_{\mathbf{V}_0}$ following from the fact that $\langle \mathbf{V}_0, \mathbf{e}_z \rangle = 0$:

- $R_{\mathbf{V}_0}^2 = \text{Id}$,
- $\alpha \geq \beta \Rightarrow R_{\mathbf{V}_0}(\alpha) \geq R_{\mathbf{V}_0}(\beta)$,
- $R_{\mathbf{V}_0} = R_{-\mathbf{V}_0}$, and
- $\alpha_{\mathbf{V}_0}^\pm = \alpha_{-\mathbf{V}_0}^\mp$.

Thus we obtain,

$$\begin{aligned} \alpha_{\mathbf{V}_0}^+ &= R_{\mathbf{V}_0}^2(\alpha_{\mathbf{V}_0}^+) = R_{\mathbf{V}_0}(R_{\mathbf{V}_0}(\alpha_{\mathbf{V}_0}^+)) \\ &\geq R_{\mathbf{V}_0}(\alpha_{\mathbf{V}_0}^-) = R_{-\mathbf{V}_0}(\alpha_{-\mathbf{V}_0}^+) \\ &\geq \alpha_{-\mathbf{V}_0}^- = \alpha_{\mathbf{V}_0}^+. \end{aligned}$$

We must have equality all the way through and hence

$$(6) \quad R_{\mathbf{V}_0}(\alpha_{\mathbf{V}_0}^-) = \alpha_{\mathbf{V}_0}^+$$

for any \mathbf{V}_0 .

To finish, equation (6) implies that α must have a horizontal (i.e. no \mathbf{e}_z component) tangent at $P_{\mathbf{V}_0} \cap \alpha$. But every point of α lies on $P_{\mathbf{V}_0}$ for some \mathbf{V}_0 hence α has a horizontal tangent everywhere and hence is a round circle. \square

Theorem 5.4. *Let γ_t be a convex, ancient solution to the curve shortening flow. Then γ_t is the unique up to isometry of \mathbb{S}^2 , family of shrinking circles ancient solution.*

Proof. The approximate symmetry preserved lemma 5.2, implies that for every \mathbf{V} with $\delta(\mathbf{V}) \in (0, \pi/4)$, $R_{\mathbf{V}}((\gamma_t)_{\mathbf{V}}^+) \geq (\gamma_t)_{\mathbf{V}}^-$ for all $t \in (-\infty, T)$. The exact symmetry proposition 5.3 applies at each such $t \in (-\infty, T)$, showing that γ_t is a round circle for every $t \in (-\infty, T)$. Uniqueness of solutions ensures that γ_t is a round circle for every $t \in (-\infty, 0)$. \square

REFERENCES

- [CG01] Bennett Chow and Robert Gulliver. “Aleksandrov reflection and geometric evolution of hypersurfaces”. In: *Comm. Anal. Geom.* 9.2 (2001), pp. 261–280.
- [CG96] Bennett Chow and Robert Gulliver. “Aleksandrov reflection and nonlinear evolution equations. I. The n -sphere and n -ball”. In: *Calc. Var. Partial Differential Equations* 4.3 (1996), pp. 249–264.

- [Cho97] Bennett Chow. “Geometric aspects of Aleksandrov reflection and gradient estimates for parabolic equations”. In: *Comm. Anal. Geom.* 5.2 (1997), pp. 389–409.
- [DHS10] Panagiotia Daskalopoulos, Richard Hamilton, and Natasa Sesum. “Classification of compact ancient solutions to the curve shortening flow”. In: *J. Differential Geom.* 84.3 (2010), pp. 455–464.
- [Gag84] M. E. Gage. “Curve shortening makes convex curves circular”. In: *Invent. Math.* 76.2 (1984), pp. 357–364.
- [GH86] M. Gage and R. S. Hamilton. “The heat equation shrinking convex plane curves”. In: *J. Differential Geom.* 23.1 (1986), pp. 69–96.
- [Gra87] Matthew A. Grayson. “The heat equation shrinks embedded plane curves to round points”. In: *J. Differential Geom.* 26.2 (1987), pp. 285–314.
- [Gra89] Matthew A. Grayson. “Shortening embedded curves”. In: *Ann. of Math. (2)* 129.1 (1989), pp. 71–111.
- [Ham95] Richard S. Hamilton. “The formation of singularities in the Ricci flow”. In: *Surveys in differential geometry, Vol. II (Cambridge, MA, 1993)*. Int. Press, Cambridge, MA, 1995, pp. 7–136.
- [Hui98] Gerhard Huisken. “A distance comparison principle for evolving curves”. In: *Asian J. Math.* 2.1 (1998), pp. 127–133.
- [JM10] David L. Johnson and Murugiah Muraleetharan. “Singularity formation of embedded curves evolving on surfaces by curvature flow”. In: *Int. J. Pure Appl. Math.* 61.2 (2010), pp. 121–146.
- [Top06] Victor Andreevich Toponogov. *Differential geometry of curves and surfaces*. A concise guide, With the editorial assistance of Vladimir Y. Rovnski. Birkhäuser Boston Inc., 2006, pp. xiv+206. ISBN: 978-0-8176-4384-3; 0-8176-4384-2.
- [Zhu98] Xi-Ping Zhu. “Asymptotic behavior of anisotropic curve flows”. In: *J. Differential Geom.* 48.2 (1998), pp. 225–274.

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